Math 122 / Problem Set 3

Written problems due Wednesday, October 12

Monday, October 3

1. Find all solutions to the congruence $3x \equiv 7$ **(a)** modulo 25 and **(b)** modulo 15.

2. Prove the well-known rule for divisibility by 9: that every positive integer is congruent to the sum of its decimal digits modulo 9. State and prove a similar rule for divisibility by 11.

3. Use Artin proposition 2.2.6 to prove the *Chinese Remainder Theorem*: Let m, n, a, b be integers, and assume that the greatest common divisor of m and n is 1. Then there is an integer x such that $x \equiv a$ modulo m and $x \equiv b$ modulo n.

Reading: Artin §2.10

Wednesday, October 5

4. Let $\{e, (12), (23), (13), (123), (132)\}$ be the presentation of S_3 considered in class. Let H be the subgroup $\{e, (12)\}$. Compute the product sets $(eH)((123)H)$ and $(eH)((132)H)$, and verify that they are not cosets.

5. Now let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB is contained entirely within another element C of the partition. Let N be the element of P which contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets.

6. Prove that the subset $G \times 1$ of the product group $G \times G'$ is a normal subgroup isomorphic to G and that $(G \times G')/(G \times 1)$ is isomorphic to G'.

7. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly, and prove that G/H is isomorphic to G.

8. Construct an isomorphism from $\mathbb{R}/\mathbb{Z}$ to $U = \{z \in \mathbb{C}^\times : |z| = 1\}$.

Reading: Artin §§3.1, 3.2

Friday, October 7

9. Which of the following subsets of the vector space of real $n \times n$ matrices is a subspace?

(a) symmetric matrices ($A = {}^tA$)
(b) invertible matrices
(c) upper triangular matrices

10. Consider the system of linear equations

$$\begin{bmatrix} 8 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

(a) Solve it in $\mathbb{F}_p$ when $p = 5, 17$.
(b) Determine the number of solutions when $p = 7$.

Reading: Artin §§3.3, 3.4, 3.6

Prepare for take-home quiz given on Tuesday, October 11 and due on Wednesday, October 12.